Lecture 5

Illustrative examples of Parallelogram Law of forces

(1.) Find the angle between the concurrent forces $\vec{P}$ and $3\vec{P}$ if their resultant is of magnitude $\sqrt{13}\,P$.

Sol. Let α be the angle between forces $\vec{P}$ and $3\vec{P}$.

Then, $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$ gives

$$(\sqrt{13}\,P)^2 = P^2 + (3P)^2 + 2P(3P)\cos\alpha$$

$$13P^2 = P^2 + 9P^2 + 6P^2\cos\alpha$$

$$3P^2 = 6P^2\cos\alpha$$

$$\Rightarrow \cos\alpha = \frac{1}{2}$$

$$\Rightarrow \alpha = 60^\circ.$$

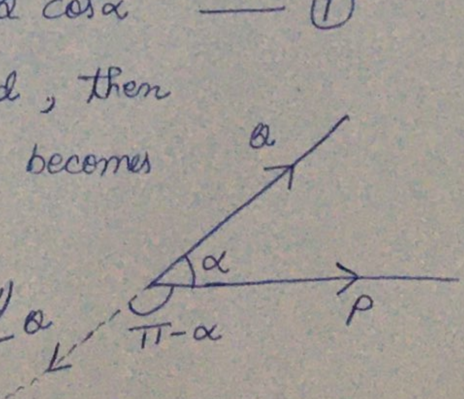
(2) The resultant of two forces $\vec{P}$ and $\vec{Q}$ acting on a particle is $\vec{R}$. If one of the forces is reversed, the resultant is $\vec{R}'$, show that

$$R^2 + R'^2 = 2(P^2+Q^2).$$

Sol. Let α be the angle between forces $\vec{P}$ and $\vec{Q}$.

Then, $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$ —— (1)

Now, if force $\vec{Q}$ is reversed, then angle between $\vec{P}$ and $-\vec{Q}$ becomes $\pi - \alpha$.



$$\therefore\ R'^2 = P^2 + (-Q)^2 + 2PQ\cos(\pi-\alpha)$$

$$R'^2 = P^2 + Q^2 - 2PQ\cos\alpha$$

—— (2)

On adding 1 and 2, we get

$$R^2 + R'^2 = 2(P^2+Q^2)$$

Hence, proved.

3 The magnitude of the resultant of two forces $\vec{P}$ and $\vec{Q}$ acting at angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$. When they act at an angle $\frac{\pi}{2}-\theta$, the resultant is of magnitude $(2m-1)\sqrt{P^2+Q^2}$. Show that

$$\tan\theta = \frac{m-1}{m+1}$$

Sol. Since $(2m+1)\sqrt{P^2+Q^2}$ is magnitude of the resultant of forces $\vec{P}$ and $\vec{Q}$ acting at angle θ, therefore,

$$\left[(2m+1)\sqrt{P^2+Q^2}\right]^2 = P^2+Q^2+2PQ\cos\theta$$

$$\Rightarrow (4m^2+4m+1)(P^2+Q^2) = P^2+Q^2+2PQ\cos\theta$$

$$\Rightarrow (4m^2+4m)(P^2+Q^2) = 2PQ\cos\theta$$

$$\Rightarrow 4m(m+1)(P^2+Q^2) = 2PQ\cos\theta \quad \text{—— } (1)$$

Again $(2m-1)\sqrt{P^2+Q^2}$ is magnitude of the resultant of forces $\vec{P}$ and $\vec{Q}$ acting at angle $\frac{\pi}{2}-\theta$, therefore

$$\left((2m-1)\sqrt{P^2+Q^2}\right)^2 = P^2+Q^2+2PQ\cos\left(\frac{\pi}{2}-\theta\right)$$

$$\Rightarrow (4m^2+1-4m)(P^2+Q^2) = P^2+Q^2+2PQ\sin\theta$$

$$\Rightarrow (4m^2-4m)(P^2+Q^2) = 2PQ\sin\theta$$

$$\Rightarrow 4m(m-1)(P^2+Q^2) = 2PQ\sin\theta \quad \text{—— } (2)$$

On dividing 2 by 1, we get

$$\frac{4m(m-1)(P^2+Q^2)}{4m(m+1)(P^2+Q^2)} = \frac{2PQ\sin\theta}{2PQ\cos\theta}$$

$$\Rightarrow \quad \frac{m-1}{m+1} = \tan\theta$$

$$\Rightarrow \quad \tan\theta = \frac{m-1}{m+1}$$

(4) The greatest and least resultants that two forces can have are of magnitude P and Q respectively. Show that when they act at angle θ, their resultant is of magnitude $\sqrt{P^2\cos^2\frac{\theta}{2} + Q^2\sin^2\frac{\theta}{2}}$

Sol. Let $\vec{R}$ and $\vec{S}$ be given forces, where $R > S$.

According to given information,

$$P = R + S$$
$$Q = R - S$$

On solving these equations, we get

$$R = \frac{P+Q}{2} \quad \text{and} \quad S = \frac{P-Q}{2}$$

Let $\vec{R'}$ be resultant of forces $\vec{R}$ and $\vec{S}$ acting at angle θ.

Then, $R'^2 = R^2 + S^2 + 2RS\cos\theta$

$$= \left(\frac{P+Q}{2}\right)^2 + \left(\frac{P-Q}{2}\right)^2 + 2\left(\frac{P+Q}{2}\right)\left(\frac{P-Q}{2}\right)\cos\theta$$

$$= \frac{1}{4}(P^2+Q^2+2PQ) + \frac{1}{4}(P^2+Q^2-2PQ) + \frac{1}{2}(P^2-Q^2)\cos\theta$$

$$= \frac{1}{4}(2P^2) + \frac{1}{4}(2Q^2) + \frac{1}{2}P^2\cos\theta - \frac{1}{2}Q^2\cos\theta$$

$$= \frac{1}{2}P^2(1+\cos\theta) + \frac{1}{2}Q^2(1-\cos\theta).$$

$$= \frac{1}{2}P^2 \cdot 2\cos^2\frac{\theta}{2} + \frac{1}{2}Q^2 \cdot 2\sin^2\frac{\theta}{2}$$

$$= P^2\cos^2\frac{\theta}{2} + Q^2\sin^2\frac{\theta}{2}$$

$$\Rightarrow \vec{R}' = \sqrt{P^2\cos^2\frac{\theta}{2} + Q^2\sin^2\frac{\theta}{2}} \, .$$